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Mathematical Note

On the Mean Curvatures Sharp Estimates of Hypersurfaces

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Abstract

Sharp estimates for the mean curvatures of hypersurfaces in Riemannian manifolds are known from the works of Jorge-Xavier [3], Markvorsen [6] and Vlachos [11]. We first give a simplified proof of these estimates. This proof shows that a similar original result holds for hypersurfaces in Einstein manifolds which are warped product of $\mathbb{R}$ by Ricci-flat manifolds.

1 Introduction and notations

For $n \geq 1$ and $c \in \mathbb{R}$, let $(\mathbb{M} = \mathbb{M}_{n+1}(c), g = \langle\cdot,\cdot\rangle)$ be the $(n+1)$- dimensional simply connected space form of constant curvature c, d its Riemannian distance, $\overline{\nabla}$ its Levi-Civita connection and $\overline{\nabla}^2$ its Hessian operator. If N is a closed (compact without boundary) connected hypersurface of $\mathbb{M}$, we endow N with the induced metric, also denoted by $\langle\cdot,\cdot\rangle$. The induced connection and Hessian are denoted by ∇ and ∇^2 respectively. By the generalized Jordan theorem, N is orientable and divides $\mathbb{M}$ into two connected components, one of which (the *interior*) is relatively compact and has N as its oriented boundary. Let η the smooth unit *inner* normal vector field of N, h its second fundamental form and A its shape operator. We recall that the mean curvatures of N are the functions $(H_i)_{0\leq i\leq n}$ defined by $\prod_{i=1}^n(1+Xk_i) = \sum_{i=0}^n \sigma_i X^i = \sum_{i=0}^n \binom{n}{i} H_i X^i$ where $(k_i)_{1\leq i\leq n}$ are the principal curvatures of N, σ_i the i^{th}-elementary symmetric polynomial in the k_i's and $\binom{n}{i}$ the binomial coefficient. The notation $\|H_i\|$ will mean the uniform norm of H_i on N. We also introduce the *radius* of N in $\mathbb{M}$, defined by $\operatorname{rad} N := \min_{p\in\mathbb{M}} \max_{q\in N} d(p,q)$ and which is the radius of the smallest closed ball(s) in $(\mathbb{M}, d)$ containing N. We recall that a geodesic sphere of radius r (with $r < \pi/\sqrt{c}$ if $c > 0$) in $\mathbb{M}$ is a totally umbilical hypersurface: more precisely, if sn_c is the solution of the differential equation $\ddot{y}(t) + cy(t) = 0$ with the initial conditions $(y(0), \dot{y}(0)) = (0, 1)$ and $\cot_c = \dot{\operatorname{sn}}_c/\operatorname{sn}_c$ its logarithmic derivative, its principal curvatures are all equal to $-\cot_c(r)$.

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Thanks to the works of Jorge-Xavier, Markvorsen and Vlachos, we have the following estimates for the mean curvatures of N and this leads to a characterization of geodesic spheres:

Theorem 1 [3, 6, 11]. *Let N be a closed connected hypersurface of $\mathbb{M}$. We assume that $rad\, N < \pi/(2\sqrt{c})$ if c is positive. Then*

i) For any integer $k \in \{1, \ldots, n\}$, we have $\| H_k \| \geq \cot_c^k (rad\, N)$.
In other words, we obtain a sharp lower bound for the radius of N:

$$rad\, N \geq \cot_c^{-1} \left(\min_{k \in \{1,\ldots,n\}} \| H_k \|^{1/k} \right) .$$

ii) If there exists an integer $k \in \{1, \ldots, n\}$ for which $\| H_k \| = \cot_c^k (rad\, N)$, then N is a geodesic sphere.

We will present here a simplified proof of this result. The key point is to choose an appropriate function and to consider the Newton (1,1)-tensors introduced by Reilly. The papers quoted above did not show the necessity of the assumption "rad $M < \pi/(2\sqrt{c})$" when c is positive: we fill this gap by producing a counter-example at the end of section 2. Moreover, our approach allows the discovering of a new similar result (section 3, theorem 3).

2 A short proof of theorem 1

Let h_c be the primitive of sn_c which vanishes at 0. For a fixed point p of $\mathbb{M}$, we introduce the smooth *modified distance function* $h_c \circ d_p$ on $\mathbb{M}$ which Hessian is proportional to the metric:

Proposition [9]. - *The function $h_c \circ d_p$ satisfies: $\overline{\nabla}^2 (h_c \circ d_p) = (\dot{\mathrm{sn}}_c \circ d_p) \cdot g$.*

In the sequel, let p be a point of $\mathbb{M}$ for which N is included in the closed ball $B_d(p, \mathrm{rad}\, N)$. If we set $F = h_c \circ d_p$, let $f = F_{|N}$ be the restriction of F to N and q_0 a point of N where f achieves its maximum. By the Hopf principle, we have $\langle \nabla f(q_0), X \rangle = 0$ and $\nabla^2 f(q_0)(X, X) \leq 0$ for any $X \in T_{q_0} N$. But

$$\langle \nabla f(q_0), X \rangle = \langle \overline{\nabla} F(q_0), X \rangle = \mathrm{sn}_c(d_p(q_0)) \cdot \langle \dot{\gamma}(d_p(q_0)), X \rangle$$

where $\gamma : [0, d_p(q_0)] \to \mathbb{M}$ is the unique unit-speed geodesic in $\mathbb{M}$ joining p to q_0. This shows that $\dot{\gamma}(d_p(q_0)) \in (T_{q_0} N)^{\perp}$. On the other side, by the above proposition

$$\begin{aligned} \nabla^2 f(q_0)(X, X) &= \overline{\nabla}^2 F(q_0)(X, X) + \langle \overline{\nabla} F(q_0), h_{q_0}(X, X) \rangle \\ &= \dot{\mathrm{sn}}_c(d_p(q_0)) \cdot |X|^2 + \langle \overline{\nabla} F(q_0), h_{q_0}(X, X) \rangle \end{aligned}$$

In this way,

$$\nabla^2 f(q_0)(X,X) \geq \dot{\mathrm{sn}}_c(d_p(q_0)) \cdot |X|^2 - \mathrm{sn}_c(d_p(q_0)) \cdot \langle A_{q_0}X, X\rangle \tag{1}$$

This shows that the principal curvatures $(k_i(q_0))_{1\leq i\leq n}$ of N at q_0 are all greater than or equal to $\cot_c(d_p(q_0))$. Since $d_p(q_0) \leq \mathrm{rad}\, N$, as $\cot_c$ is a decreasing function and as $\cot_c(\mathrm{rad}\, N)$ is positive (we use here the assumption on the radius if c is positive), we have $\|H_k\| \geq \cot_c^k(\mathrm{rad}\, N)$ and this shows the first point of the theorem.

If there is equality, i.e. if $\|H_k\| = \cot_c^k(\mathrm{rad}\, N)$, then $d_p(q_0) = \mathrm{rad}\, N$ and all the principal curvatures of N at q_0 are equal to $\cot_c(\mathrm{rad}\, N)$ which is positive. Let $\mathcal{U}$ be an open neighborhood of q_0 in N such that the principal curvatures are all positive on $\mathcal{U}$. On $\mathcal{U}$, we will use the classical inequalities [2]:

$$H_1 \geq H_2^{1/2} \geq \cdots \geq H_{k-1}^{1/(k-1)} \geq H_k^{1/k} \geq \cdots \geq H_n^{1/n}$$

In [7], Reilly introduced a family $(T_k)_{k\in\{0,\dots,n\}}$ of (1,1)-tensors on N defined by the formulae: $T_0 = Id$ (identity map) and $T_{k+1} = \sigma_{k+1}Id - AT_k$ for $0 \leq k \leq n-1$ which satisfy the following formula :

$$\mathrm{Div}(T_k\nabla f) = (n-k)\cdot\binom{n}{k}\cdot\left\{(\dot{\mathrm{sn}}_c \circ d_p)\cdot H_k + \langle\overline{\nabla}F, \eta\rangle\cdot H_{k+1}\right\} \tag{2}$$

where Div is the divergence operator on N. As $\cot_c$ is a decreasing function, then for any point q of $\mathcal{U}$, we have by equation (2):

$$\begin{aligned}
\frac{\mathrm{Div}\ (T_{k-1}\nabla f)(q)}{(n-k+1)\cdot\binom{n}{k-1}} &= \dot{\mathrm{sn}}_c(d_p(q))\cdot H_{k-1}(q) + \langle\overline{\nabla}F(q), \eta(q)\rangle\cdot H_k(q)\\
&\geq \dot{\mathrm{sn}}_c(d_p(q))\cdot H_{k-1}(q) - \mathrm{sn}_c(d_p(q))\cdot H_k(q)\\
&= \mathrm{sn}_c(d_p(q))\cdot\{\cot_c(d_p(q))\cdot H_{k-1}(q) - H_k(q)\}\\
&\geq \mathrm{sn}_c(d_p(q))\cdot\{\cot_c(d_p(q_0))\cdot H_{k-1}(q) - H_k(q)\}\\
&= \mathrm{sn}_c(d_p(q))\cdot\{\|H_k\|^{1/k}\cdot H_{k-1}(q) - H_k(q)\}
\end{aligned}$$

Now $H_k(q) = H_k^{1/k}(q)\cdot H_k^{(k-1)/k}(q) \leq \|H_k\|^{1/k}\cdot H_{k-1}(q)$. This shows that Div $(T_{k-1}\nabla f)$ is nonnegative on $\mathcal{U}$. As $(\cdot \mapsto \mathrm{Div}\ (T_{k-1}\nabla\cdot))$ is an elliptic operator on smooth functions on $\mathcal{U}$ ([4]), the function f is therefore constant on $\mathcal{U}$ by the maximum principle for these operators. Hence, the non-empty closed subset $\{q \in N / f(q) = f(q_0)\}$ of N is also open. The connectedness of N implies that N is included in the geodesic sphere $F_p^{-1}(F_p(q_0))$. As this geodesic sphere is also connected and n-dimensional, N coincides with this geodesic sphere. □

Remark 1. For $k = 1$, theorem 1 can be stated if one replaces $\mathbb{M}_{n+1}(c)$ by a manifold M with sectional curvature bounded from above by c: this has been done by Markvorsen [6], Jorge-Xavier [3] and can be derived easily from the above equations: indeed, for a hypersurface in an arbitrary manifold, (2) is still true for $k = 0$. On the other hand, we have the following comparison result:

Lemma ([8], p 153). - *Let M be a complete Riemannian manifold with sectional curvature bounded from above by a constant c ($c \in \mathbb{R}$), d the distance of M and p a point of M. Then if q_0 is not a cut point of p, the function d_p is smooth at q_0 and for any vector $X \in T_{q_0}M$ which is normal at q_0 to the unique unit-speed geodesic joining p to q_0, we have*

$$\overline{\nabla}^2 d_p(q_0)(X,X) \geq \cot_c(d_p(q_0)) \cdot |X|^2 .$$

which makes true inequation (1). The end of proof is similar.

Remark 2: a counter-example without the radius assumption. Let n be an integer ≥ 2, c a positive number, j and k two integers ≥ 1 with $j+k=n$ and s a number of $]0,\pi/2[$. We will write $\mathbb{R}^{n+2} = \mathbb{R}^{j+1} \times \mathbb{R}^{k+1}$ and any point x of $\mathbb{R}^{n+2}$ will be decomposed as $x=(y,z)$ where $(y,z) \in \mathbb{R}^{j+1} \times \mathbb{R}^{k+1}$. In [10], the author proves that $N := \mathbb{S}^j(\cos(s)/\sqrt{c}) \times \mathbb{S}^k(\sin(s)/\sqrt{c})$ is a compact connected hypersurface of $\mathbb{M}_{n+1}(c) = \mathbb{S}^{n+1}(1/\sqrt{c}) = \{x=(y,z) \in \mathbb{R}^{j+1} \times \mathbb{R}^{k+1} / \ |y|^2+|z|^2 = 1/c\}$ which principal curvatures at any point are $(-\sqrt{c}\tan s)$ et $\sqrt{c}\cot s$ with multiplicities j and k respectively (for c=j=k=1 and $s=\pi/4$, one recognizes the Clifford torus in $\mathbb{S}^3$).

Moreover, We claim that

$$\operatorname{rad} N = \frac{1}{\sqrt{c}} \cdot \left\{ \frac{3\pi}{4} - \left| s - \frac{\pi}{4} \right| \right\} .$$

which proof is a straightforward calculation: let $p=(y_p,z_p)$ and $q=(y,z)$ be arbitrary points of $\mathbb{S}^{n+1}(1/\sqrt{c})$ and N respectively. In $\mathbb{S}^{n+1}(1/\sqrt{c})$, the distance between p and q is $d(p,q) = (1/\sqrt{c}) \cdot \cos^{-1}(c\langle p,q\rangle)$. By Cauchy-Schwarz inequality, one obtains $c\langle p,q\rangle = c(\langle y_p \cdot y\rangle + \langle z_p \cdot z\rangle) \geq -c(|y_p|\cdot|y|+|z_p|\cdot|z|) = -\sqrt{c}(|y_p|\cdot\cos s + |z_p|\cdot\sin s)$. So $\max_{q\in N} d(p,q) \leq (1/\sqrt{c})\cdot\cos^{-1}\{-\sqrt{c}(|y_p|\cdot\cos s + |z_p|\cdot\sin s)\}$. Moreover, this inequality is sharp (indeed, if y_p and z_p are both non zero, take $(y,z) = (-(y_p/|y_p|)\cos s, -(z_p/|z_p|)\sin s)$ and if $y_p=0$, then z_p is nonzero necessarily and take $z = -(z_p/|z_p|)\sin s$ and any point of $\mathbb{S}^j(\cos(s)/\sqrt{c})$ for y). Using the relation $\cos^{-1} a + \cos^{-1}(-a) = \pi$, we deduce that

$$\begin{aligned}
&\operatorname{rad} N \\
&= \min_{p\in\mathbb{S}^{n+1}(1/\sqrt{c})}\{\pi/\sqrt{c} - \cos^{-1}\{\sqrt{c}\cdot(|y_p|\cdot\cos s + |z_p|\cdot\sin s)\}\} \\
&= \pi/\sqrt{c} - (1/\sqrt{c})\cos^{-1}\{\min_{p\in\mathbb{S}^{n+1}(1/\sqrt{c})}\{\sqrt{c}\cdot(|y_p|\cdot\cos s + |z_p|\cdot\sin s)\}\} \\
&= \pi/\sqrt{c} - (1/\sqrt{c})\cos^{-1}\{\min_{0\leq|y_p|\leq(1/\sqrt{c})}\{\sqrt{c}\cdot(|y_p|\cdot\cos s + \\
&\qquad\qquad\qquad\qquad \sqrt{1/c - |y_p|^2}\cdot\sin s)\}\} \\
&= \pi/\sqrt{c} - (1/\sqrt{c})\cos^{-1}\min\{\cos s, \sin s\} \\
&= \pi/\sqrt{c} - (1/\sqrt{c})\max\{s, \pi/2 - s\} \\
&= (1/\sqrt{c})\cdot\{3\pi/4 - |s-\pi/4|\}. \qquad \Box
\end{aligned}$$

In the particular case where $s=\pi/4$ and $j=k=n/2$, the radius of N is $3\pi/(4\sqrt{c})$ and $\cot_c(\operatorname{rad} N) = -\sqrt{c}$. Since the mean curvatures of N satisfy the relation $\sum_{i=0}^{n}\binom{n}{i}\cdot H_i\, X^i =$

$(1 - X\sqrt{c})^j \cdot (1 + X\sqrt{c})^j = (1 - cX^2)^j = \sum_{\ell=0}^{j} (-c)^\ell \binom{j}{\ell} \cdot X^{2\ell}$, the mean curvatures of odd order all vanish and

$$\text{for } \ell = 1, \dots, j, \qquad \|H_{2\ell}\| = \frac{\binom{j}{\ell}}{\binom{2j}{2\ell}} \cdot c^\ell \begin{cases} = & \cot_c^{2\ell}(\operatorname{rad} N) \quad if \;\; \ell = j \\ < & \cot_c^{2\ell}(\operatorname{rad} N) \quad if \;\; \ell < j \end{cases} .$$

As N is not a geodesic sphere (not even homeomorphic), this shows that the radius assumption cannot be omitted in theorem 1.

3 A New result

The Hessian of D is proportional to the metric g of $\mathbb{M}_{n+1}(c)$. This remark has simplified a lot the calculation of $\nabla^2 f$ in section 2. We are naturally led to ask for natural questions:

Question 1. *"Which are the complete Riemannian manifolds $(M, g, \overline{\nabla})$ admitting a smooth function F which Hessian satisfy $\overline{\nabla}^2 F = \lambda g$ for some function λ ?*

Question 2. *"Among them, which ones admit totally umbilical hypersurfaces ?"*

Other manifolds than space forms satisfying this both questions exist:

Example. Consider a n-dimensional complete Ricci-flat manifold $(N_\star, g_\star)$ and consider the Riemannian manifold $M = \mathbb{R} \times_{e^{2ct}} N_\star$ with the warped product metric $g = dt^2 + e^{2ct} \cdot g_\star$ where c is a constant. The function $F : M \to \mathbb{R} : (t, x) \mapsto e^{ct}$ satisfies $\overline{\nabla}^2 F = c^2 F \cdot g$ and the levels $(\{t\} \times N_\star)_{t \in \mathbb{R}}$ of F are totally umbilical hypersurfaces of M with principal curvatures all equal to $-c$. We also remark that the Ricci formulae for warped products, which may be found in Besse book [1], show that $M = \mathbb{R} \times_{e^{2ct}} N_\star$ is an Einstein manifold with (constant) scalar curvature $-n(n+1)c^2$.

Fortunately, question 1 has been studied since 1925 ([5]) and solved: the manifold has to be conformally diffeomorphic to either a space form either the Riemannian product $I \times N_\star$ of an open interval I of $\mathbb{R}$ by an arbitrary n-dimensional complete manifold. If one considers only Einstein manifolds, the second question is also settled:

Theorem 2 [5]. *Let (M, g) be an $(n+1)$-dimensional complete connected Einstein manifold admitting a smooth function F which Hessian satisfies $\overline{\nabla}^2 F = \lambda g$ where λ is non-identically zero. Then M is isometric to a space form or the above example.*
Moreover, if c is the Einstein constant (i.e. the constant for which the Ricci curvature $\overline{\operatorname{Ric}}$ of (M, g) satisfies $\overline{\operatorname{Ric}} = ncg$), there exists constants s and t such that $\lambda = -cF + s$ and $|\overline{\nabla} F|^2 = -cF^2 + 2sF + t$. In particular, λ and $|\overline{\nabla} F|$ are constant on the level sets of F. At last, the non empty level sets of F above regular values are totally umbilical hypersurfaces of M, with principal curvatures all equal to $-\lambda / |\overline{\nabla} F|$.

So we are naturally led to expect a similar result to theorem 1 with hypersurfaces of $\mathbb{R} \times_{e^{2ct}} N_\star$. This is done below:

Theorem 3 . *Let N be a closed connected hypersurface of $M = \mathbb{R} \times_{e^{2ct}} N_\star$ with the warped product metric $g = dt^2 + e^{2ct} \cdot g_\star$, $(N_\star, g_\star)$ being an n-dimensional compact connected Ricci-flat manifold ($c > 0$). Then*

i) For any integer $k \in \{1, \ldots, n\}$, we have $\| H_k \| \geq c^k$.

ii) If $\| H_1 \| = c$ or $\| H_2 \| = c^2$, then $N = \{t\} \times N_\star$ for some real t and is a totally umbilical hypersurface of M with principal curvatures all equal to $-c$.

We refer again the reader to [1] for numerous examples of compact Ricci-flat manifolds. The proof of the above result is quite similar and only sketched: we apply the Hopf principle to the function $f = F_{|N}$ and obtain

$$\nabla^2 f(q_0)(X, X) \geq c^2 F(q_0) \cdot |X|^2 - cF(q_0) \cdot \langle A_{q_0} X, X \rangle$$

which shows the first part of theorem 3. To study the equality case, we claim that formula (1) is still true for $k = 0$ and $k = 1$: indeed, it is a straightforward calculation for $k = 0$. For $k = 1$, an examination of Reilly's proof shows that

$$\mathrm{Div}(T_1 \nabla f) = n(n-1) \left\{ (\mathrm{s\dot{i}n}_c \circ d_p) \cdot H_1 + \langle \overline{\nabla} F, \eta \rangle \cdot H_2 \right\} + \sum_{i=1}^{n} \langle \nabla f, (\nabla_{e_i} T_1) e_i \rangle$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal basis of N. In space forms, the Codazzi equation implies that T_1 is divergence-free that is $\sum_{i=1}^n (\nabla_{e_i} T_1) e_i = 0$. It is still zero in the present case: fix a point x in N and let us choose an orthonormal basis $\{e_i\}_{i=1}^n$ with $\nabla_{e_i} e_j(x) = 0$ for all i and j. Denoting by $\overline{R}$ the Riemann tensor of M, by $\overline{\mathrm{Ric}}$ its Ricci curvature, using Codazzi equation and Bianchi identities, we have at the point x

$$\begin{aligned}
\textstyle\sum_{i=1}^n (\nabla_{e_i} T_1) e_i &= \textstyle\sum_{i=1}^n (\nabla_{e_i} (\sigma_1 Id) - (\nabla_{e_i} A) e_i) \\
&= \nabla \sigma_1 - \textstyle\sum_{i=1}^n (\nabla_{e_i} A) e_i \\
&= \nabla \sigma_1 - \textstyle\sum_{i,j=1}^n \langle (\nabla_{e_i} A) e_i, e_j \rangle e_j \\
&= \nabla \sigma_1 - \textstyle\sum_{i,j=1}^n \langle e_i, (\nabla_{e_i} A) e_j \rangle e_j \\
&= \nabla \sigma_1 - \textstyle\sum_{i,j=1}^n \langle e_i, (\nabla_{e_j} A) e_i + \overline{R}(e_i, e_j) \eta \rangle e_j \\
&= \textstyle\sum_{i,j=1}^n \langle e_i, \overline{R}(e_i, e_j) \eta \rangle e_j \\
&= \textstyle\sum_{j=1}^n \overline{\mathrm{Ric}}(e_j, \eta) e_j \\
&= 0
\end{aligned}$$

In the case of equality, the same argument implies that $\Delta f(q) \geq ncF(q) \{ \| H_1 \| - H_1(q) \} \geq 0$ or $\mathrm{Div}(T_1 \nabla f) \geq n(n-1) cF(q) \{ \| H_2 \|^{1/2} H_1(q) - H_2(q) \} \geq 0$ on $\mathcal{U}$ and we conclude as above. □

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